
CS482: Monte Carlo Integration

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(윤성의)

<http://sglab.kaist.ac.kr/~sungeui/ICG>

KAIST

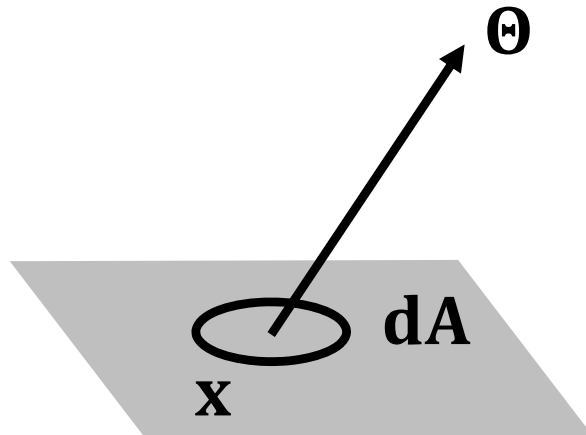
The KAIST logo consists of the word "KAIST" in a bold, blue, sans-serif font. Below the text is a light blue, horizontal oval shape that serves as a shadow or base for the letters.

Class Objectives (Ch. 14)

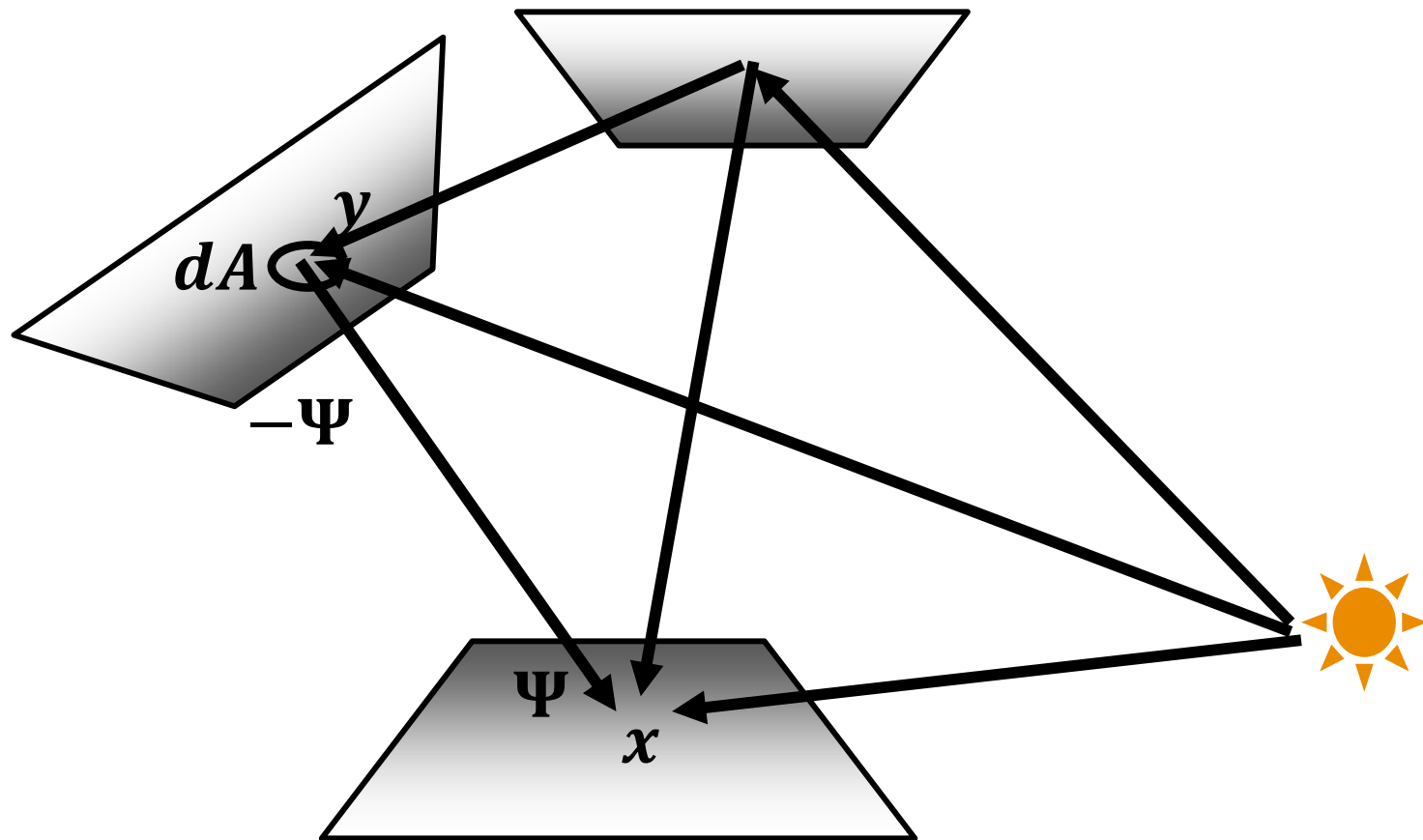
- **Sampling approach for solving the rendering equation**
 - Monte Carlo integration
 - Estimator and its variance
- **Book:**
 - <https://sgvr.kaist.ac.kr/~sungeui/render/>

Radiance Evaluation

- **Fundamental problem in GI algorithm**
 - Evaluate radiance at a given surface point in a given direction
 - Invariance defines radiance everywhere else



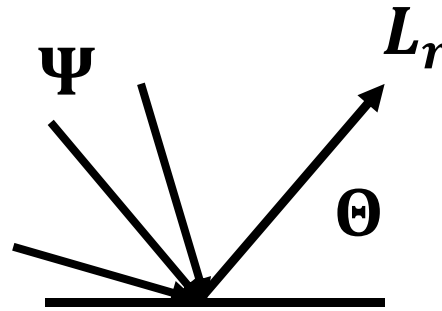
We need to find many paths...



Why Monte Carlo?

- Radiance is hard to evaluate

$$L_r(x \rightarrow \Theta) = \int_{\Psi} L(x \leftarrow \Psi) f_r(x, \Psi \rightarrow \Theta) \cos \theta_x dw_{\Psi},$$



- Sample many paths
 - Integrate over all incoming directions
- Analytical integration is difficult
 - Need numerical techniques

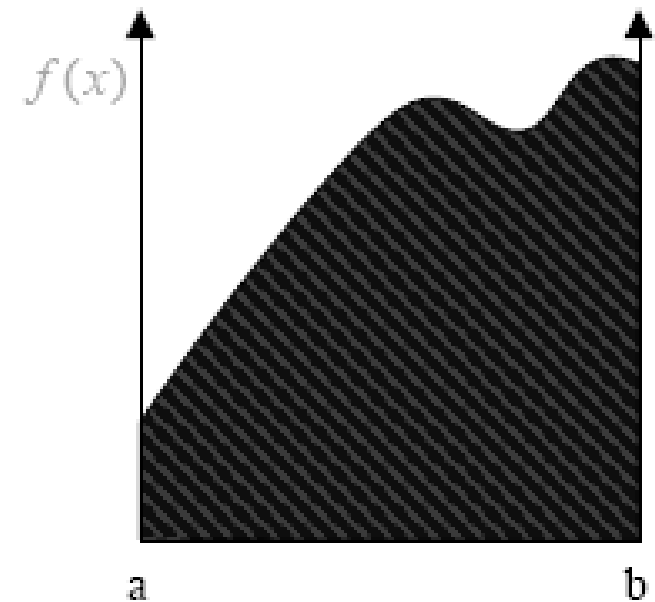
Monte Carlo Integration

- **Numerical tool to evaluate integrals**
 - Use sampling
- **Stochastic errors**
- **Unbiased**
 - On average, we get the right answer

Numerical Integration

- A one-dimensional integral:

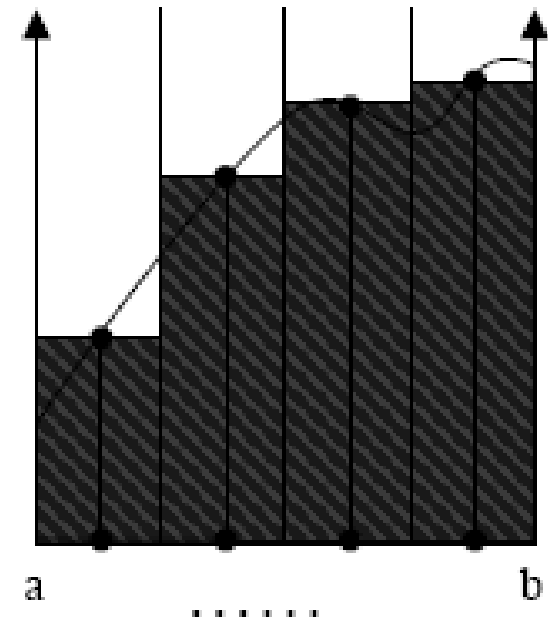
$$I = \int_a^b f(x) dx$$



Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x) dx$$
$$\approx \sum_{i=1}^N w_i f(x_i)$$

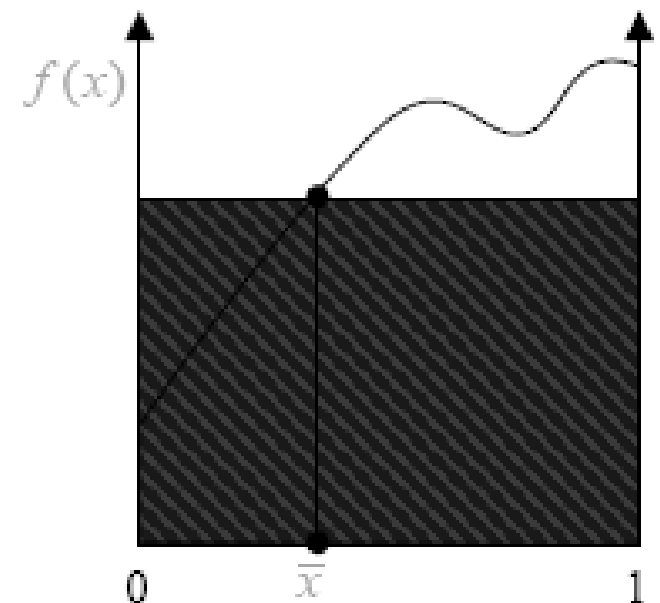


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(\bar{x})$$

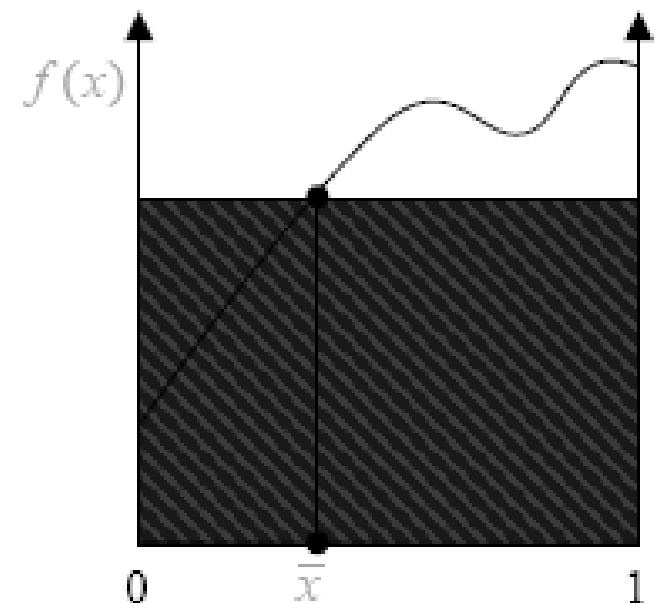


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(\bar{x})$$



$$E(I_{prim}) = \int_0^1 f(x) p(x) dx = \int_0^1 f(x) 1 dx = I$$

Unbiased estimator!

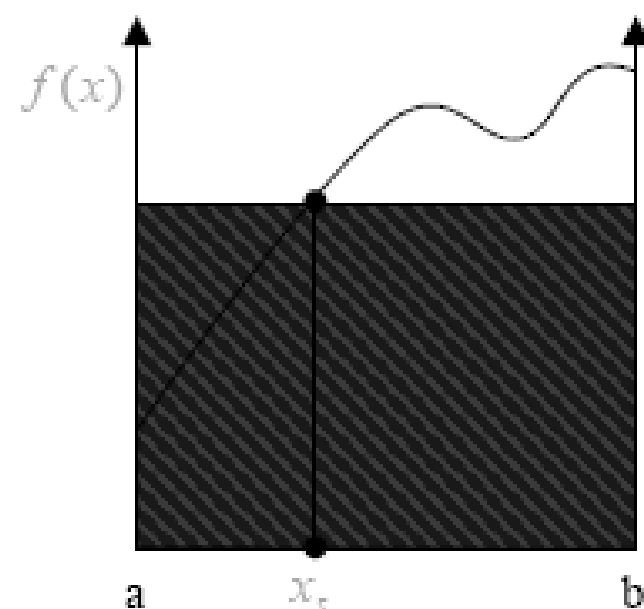


Monte Carlo Integration

Primary estimator:

$$I = \int_a^b f(x) dx$$

$$I_{prim} = f(x_s)(b - a)$$



$$E(I_{prim}) = \int_a^b f(x)(b-a)p(x)dx = \int_a^b f(x)(b-a)\frac{1}{(b-a)}dx = I$$

Unbiased estimator!



Monte Carlo Integration: Error

Variance of the estimator → a measure of the stochastic error

$$\sigma_{prim}^2 = \int_a^b \left[\frac{f(x)}{p(x)} - I \right]^2 p(x) dx$$

- 
- Consider $p(x)$ for estimate
 - We will study it as importance sampling later



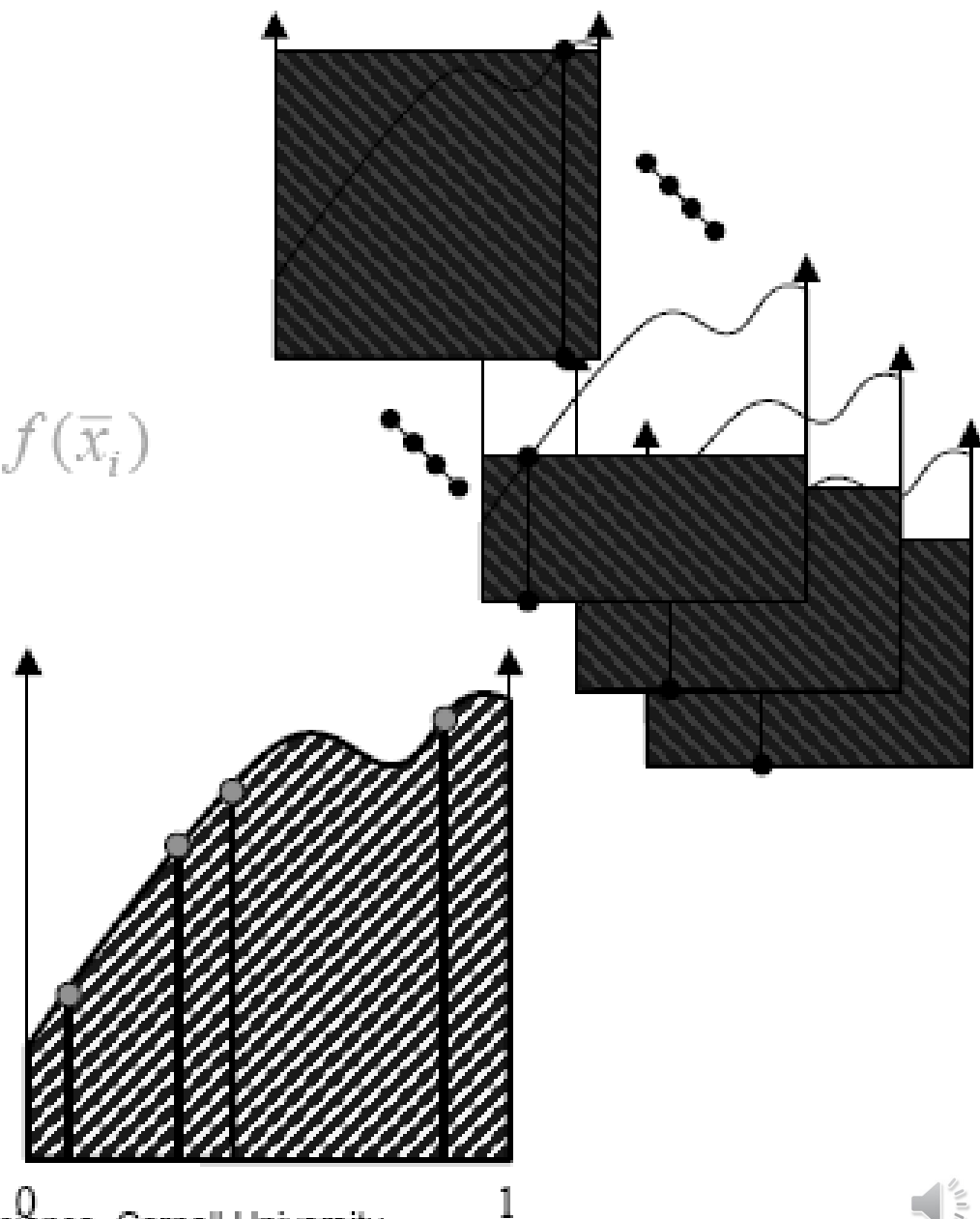
More samples

Secondary estimator

Generate N random samples $\mathbf{x}_i$

Estimator: $\langle I \rangle = I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^N f(\bar{\mathbf{x}}_i)$

Variance $\sigma_{\text{sec}}^2 = \sigma_{\text{prim}}^2 / N$



Mean Square Error of MC Estimator

- **MSE**

$$MSE(\hat{Y}) = E[(\hat{Y} - Y)^2] = \frac{1}{N} \sum_i (\hat{Y}_i - Y_i)^2.$$

- **Decomposed into bias and variance terms**

$$\begin{aligned} MSE(\hat{Y}) &= E \left[(\hat{Y} - E[\hat{Y}])^2 \right] + (E(\hat{Y}) - Y)^2 \\ &= Var(\hat{Y}) + Bias(\hat{Y}, Y)^2. \end{aligned}$$

- **Bias: how far the estimation is away from the ground truth**
- **Variance: how far the estimation is away from its average estimator**

Bias of MC Estimator

$$\begin{aligned} E[\hat{I}] &= E \left[\frac{1}{N} \sum_i \frac{f(x_i)}{p(x_i)} \right] \\ &= \frac{1}{N} \int \sum_i \frac{f(x_i)}{p(x_i)} p(x) dx \\ &= \frac{1}{N} \sum_i \int \frac{f(x)}{p(x)} p(x) dx, \because x_i \text{ samples have the same } p(x) \\ &= \frac{N}{N} \int f(x) dx = I. \end{aligned} \tag{14.6}$$

- **On average, it gives the right answer: unbiased**

Variance of MC Estimator

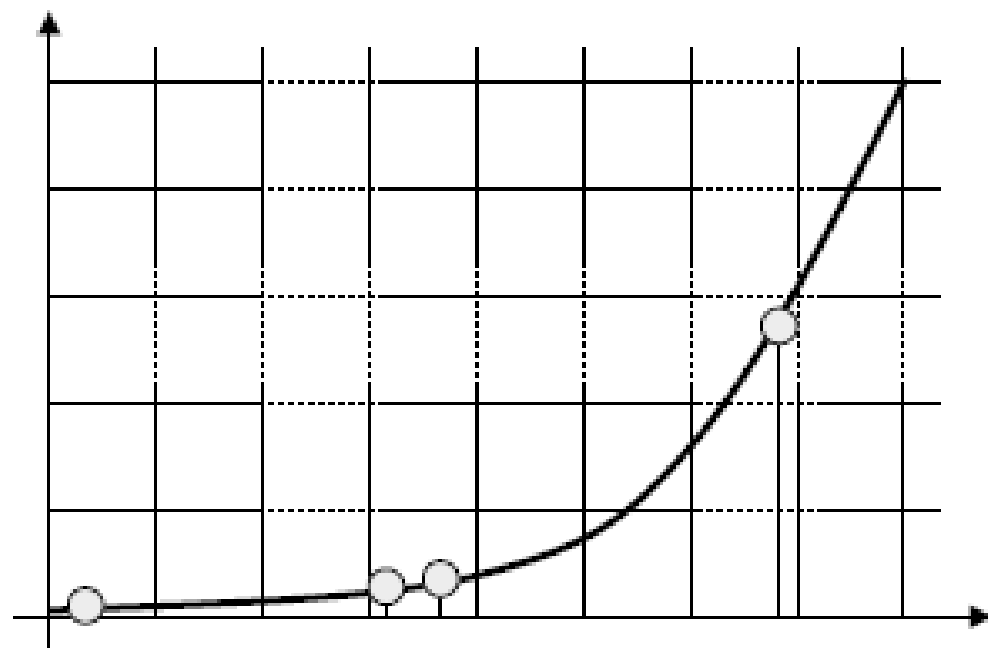
$$\begin{aligned} \text{Var}(\hat{I}) &= \text{Var}\left(\frac{1}{N} \sum_i \frac{f(x_i)}{p(x_i)}\right) \\ &= \frac{1}{N^2} \text{Var}\left(\sum_i \frac{f(x_i)}{p(x_i)}\right) \\ &= \frac{1}{N^2} \sum_i \text{Var}\left(\frac{f(x_i)}{p(x_i)}\right), \because x_i \text{ samples are independent from each other.} \\ &= \frac{1}{N^2} N \text{Var}\left(\frac{f(x)}{p(x)}\right), \because x_i \text{ samples are from the same distribution.} \\ &= \frac{1}{N} \text{Var}\left(\frac{f(x)}{p(x)}\right) = \frac{1}{N} \int \left(\frac{f(x)}{p(x)} - E\left[\frac{f(x)}{p(x)}\right]\right)^2 p(x) dx. \quad (14.7) \end{aligned}$$

MC Integration - Example

– Integral $I = \int_0^1 5x^4 dx = 1$

– Uniform sampling

– Samples :



$$x_1 = .86$$

$$\langle I \rangle = 2.74$$

$$x_2 = .41$$

$$\langle I \rangle = 1.44$$

$$x_3 = .02$$

$$\langle I \rangle = 0.96$$

$$x_4 = .38$$

$$\langle I \rangle = 0.75$$



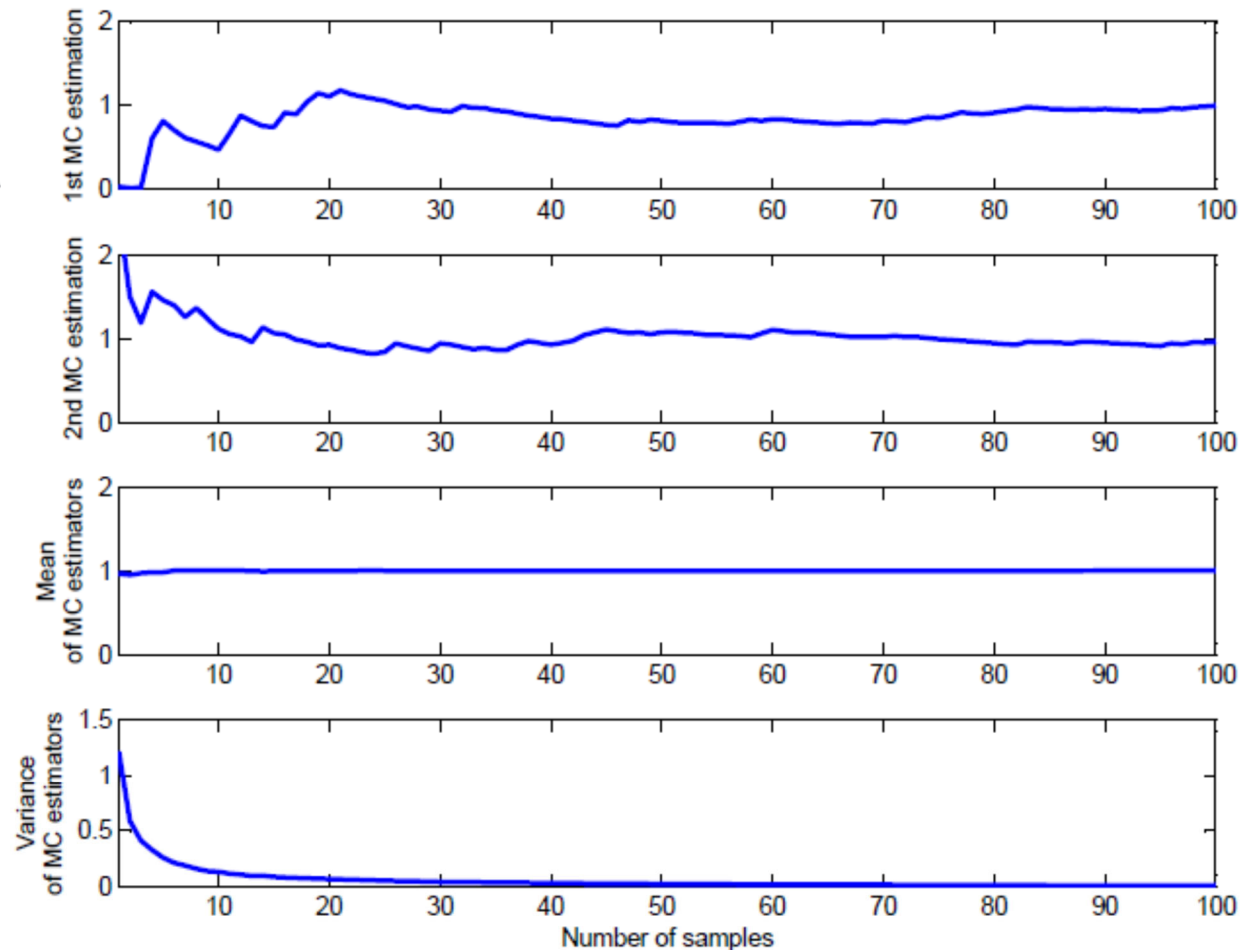
MC Integration - Example

- Integral

$$I = \int_0^1 4x^3 dx = 1$$

$$\hat{I} = \frac{1}{N} \sum_{i=1}^N 4x_i^3,$$

Code:
mc_int_ex.m

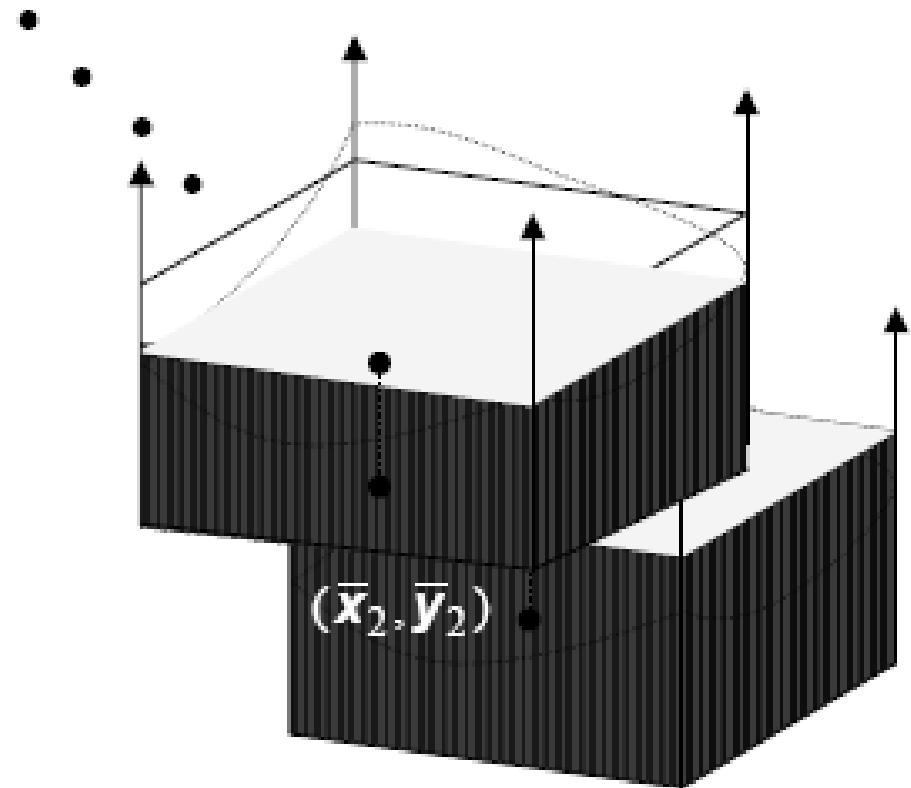


Available at my rendering book homepage

MC Integration: 2D

- Secondary estimator:

$$I_{\text{sec}} = \frac{1}{N} \sum_{i=1}^N \frac{f(\bar{x}_i, \bar{y}_i)}{p(\bar{x}_i, \bar{y}_i)}$$

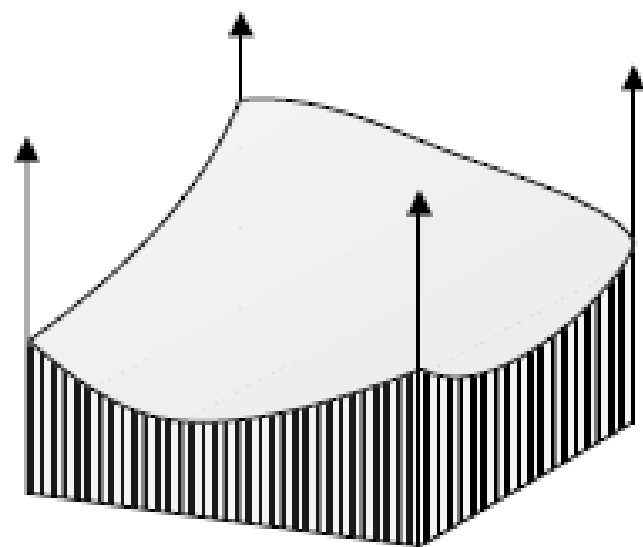


Monte Carlo Integration - 2D

- MC Integration works well for higher dimensions
- Unlike quadrature

$$I = \int_a^b \int_c^d f(x, y) dx dy$$

$$\langle I \rangle = \frac{1}{N} \sum_{i=1}^N \frac{f(x_i, y_i)}{p(x_i, y_i)}$$

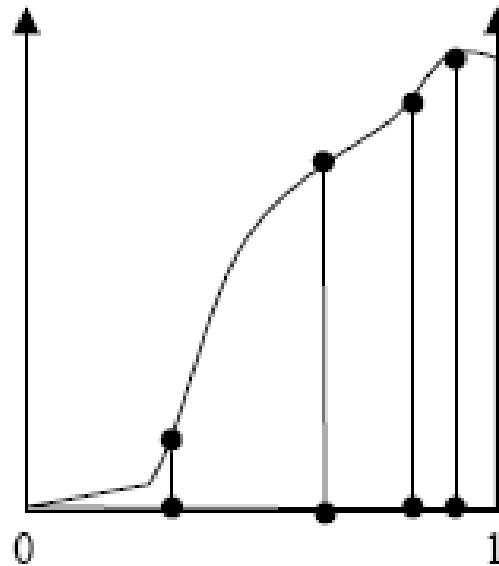


Advantages of MC

- **Convergence rate of $O(\frac{1}{\sqrt{N}})$**
- **Simple**
 - **Sampling**
 - **Point evaluation**
- **General**
 - **Works for high dimensions**
 - **Deals with discontinuities, crazy functions, etc.**

Importance Sampling

- Take more samples in important regions, where the function is large



From kavita's slides

Class Objectives (Ch. 14) were:

- **Sampling approach for solving the rendering equation**
 - **Monte Carlo integration**
 - **Estimator and its variance**

Next Time...

- **Monte Carlo ray tracing**

Homework

- **Go over the next lecture slides before the class**
- **Watch 2 top-tier papers/videos and submit your summaries before every Mon. class**
 - **Just one paragraph for each summary**

Example:

Title: XXX XXXX XXXX

Abstract: this video is about accelerating the performance of ray tracing. To achieve its goal, they design a new technique for reordering rays, since by doing so, they can improve the ray coherence and thus improve the overall performance.

Any Questions?

- **Submit four times in Sep./Oct.**
- **Come up with one question on what we have discussed in the class and submit at the end of the class**
 - **1 for typical questions**
 - **2 for questions that have some thoughts or surprise me**